### Discrete Fourier Transform

- **FS (Fourier Series)**
  - All Periodic Waves Can be Generated by Combining Sin and Cos Waves of Different Frequencies
  - DFT (Discrete FT)
    - The DFT pair
      \[ x[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \]
      \[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{j2\pi nk/N} \]
  - Complexity in terms of real operations
    - \( 4N \) real multiplications
    - \( 2N(N-1) \) real additions
  - Most fast methods are based on symmetry properties
    - Conjugate symmetry
    - Periodicity in \( n \) and \( k \)
    - FFT (Fast Fourier Transform)
      - FFT is a very efficient algorithm for performing a DFT.
      - FFT algorithm published by Cooley & Tukey in 1965.
      - In 1969, the 2048 point analysis of a seismic trace took 13½ hours.
      - Using the FFT, the same task on the same machine took 2.4 seconds!

### Decimation-In-Time FFT Algorithms

- Makes use of both symmetry and periodicity
  - Consider special case of \( N \) an integer power of 2
  - Separate \( x[n] \) into two sequence of length \( N/2 \)
    - Even indexed samples in the first sequence
    - Odd indexed samples in the other sequence
  - Substitute variables \( n=2r \) for \( n \) even and \( n=2r+1 \) for odd
    - \( X[k] = \sum_{n=0}^{N/2-1} x[2r] e^{-j2\pi nk/N} + \sum_{n=0}^{N/2-1} x[2r+1] e^{j2\pi nk/N} \)
  - Substitute variables \( n=2r \) for \( n \) even and \( n=2r+1 \) for odd
    - \( X[k] = \sum_{n=0}^{N/2-1} x[2r] W^{2r} + \sum_{n=0}^{N/2-1} x[2r+1] W^{2r+1} \)
    - \( G[k] = W^k \)
  - \( G(k) \) and \( H(k) \) are the \( N/2 \)-point DFTs of each subsequence
  - Complexity
    - \( \log_2 N \) complex multiplications and additions
  - Bit reversed indexing
    - \( x[001] = x[00] \)
    - \( x[011] = x[10] \)
8-point DFT example

Iterative-FFT Code

```plaintext
BIT-REVERSE-COPY(a, A)

ITERATIVE-FFT

BIT-REVERSE-COPY(a, A)

ITERATIVE-FFT

Nyquist frequency

• The Nyquist frequency is equal to one-half of the sampling frequency.

Sampling rate = 256 samples/second
Sampling duration = 1 second

Example

• Cos function

• Sinc function

• Gamma function

The Nyquist frequency is equal to one-half of the sampling frequency.

x(t) = 5sin(2πt)
Amplitude = 5
Frequency = 4 Hz
Effect of changing sample rate

Measuring multiple frequencies